

Amnestically induced persistence in random walks

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We study how the Hurst exponent α depends on the fraction f of the total time t remembered by non-Markovian random walkers that recall only the distant past. We find that otherwise nonpersistent random walkers switch to persistent behavior when inflicted with significant memory loss. Such memory losses induce the probability density function of the walker's position to undergo a transition from Gaussian to non-Gaussian. We interpret these findings of persistence in terms of a breakdown of self-regulation mechanisms and discuss their possible relevance to some of the burdensome behavioral and psychological symptoms of Alzheimer's disease and other dementias.

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A classic problem in physics concerns normal versus anomalous diffusion [1, 2, 3, 4]. Fractal analysis [5, 6] of random walks [1, 2, 3, 4] with memory aims at quantitatively describing the complex phenomenology observed in economic [1, 7], sociological [8], ecological [9, 10, 11], biological [2, 12], physiological [2, 13] and physical [2, 3, 4, 6, 14] systems. Markov processes exhaustively account for random walks with short-range memory. In contrast, long-range memory typically gives rise to non-Markovian walks [2, 6, 7, 15, 16, 17, 18]. The most extreme case of a non-Markovian random walk corresponds to a stochastic process with dependence on the entire history of the system. With the aim of capturing the essential dynamics of memory loss in complex systems, we investigate an idealized model for the limiting case of unbounded memory random walks with dependence on the complete [15] or partial [16] history of a binary decision process. Persistent random walkers tend to repeat past behavior, hence a plausible assumption holds that loss of memory of the past cannot cause persistence but rather can only diminish it. Moreover, all Markovian and most non-Markovian random walk models that attempt to account for persistent behavior inevitably assume a memory of the recent past, with memory loss limited to the distant past. Aiming to question such assumptions, we show here that loss of memory of the recent rather than of the distant past can actually induce persistence.

An important global property of random walks, the Hurst exponent α [17], relates to how persistently the walkers diffuse. For the case of zero drift velocity, the Hurst exponent quantifies how the mean squared displacement scales with time t :

$$\langle x^2 \rangle \sim t^{2\alpha} . \quad (1)$$

The dynamics of the random walker can range from subdiffusion ($\alpha < 1/2$), through normal diffusion ($\alpha = 1/2$) to superdiffusion ($\alpha > 1/2$), the latter characterized by persistence (i.e., long-range correlations) in the random walk. Persistent random walkers on average repeat past

behavior.

We first describe the case without memory loss [15, 16]. The random walk starts at the origin at time $t_0 = 0$ and retains memory of its complete history. At each time step the random walker moves either one step to the right or left: $x_{t+1} = x_t + v_{t+1}$ where the velocity $v_{t+1} = \pm 1$ represents a stochastic noise with two-point autocorrelations (i.e., memory). The walker can remember the entire history of prior random walk step directions $\{v_{t'}\}$ for $t' \leq t$. At time t , we randomly choose a previous time $1 \leq t' < t$ with equal *a priori* probabilities. We then choose the current step direction v_t based on the value of $v_{t'}$ in the following manner:

$$v_t = \begin{cases} +v_{t'} & \text{with probability } p \\ -v_{t'} & \text{with probability } 1 - p \end{cases} \quad (2)$$

Without loss of generality we assume that the first step always goes to the right, i.e. $v_1 = +1$. The position at time t thus follows $x_t = \sum_{t'=1}^t v_{t'}$.

The advantage of this choice of non-Markovian random walk model stems from its known exact analytical solution [15]. The probability density function (PDF) evolves according to a Gaussian propagator with a diffusion constant that depends on time t and p :

$$P(x, t) = \frac{1}{\sqrt{4\pi t D(t)}} \exp \left[-\frac{(x - \langle x(t) \rangle)^2}{4t D(t)} \right] \quad (3)$$

$$D(t, p) = \frac{1}{8p - 6} \left[\left(\frac{t}{t_0} \right)^{4p-3} - 1 \right] . \quad (4)$$

Asymptotically, the model presents nonpersistent behavior ($\alpha = 1/2$) for $p < 3/4$ and a persistence regime ($\alpha = 2p - 1$) for $p > 3/4$ (with marginal persistence for $p = 3/4$) [15, 16]. The mean displacement scales as $\langle x \rangle \sim t^{2p-1}$, decaying algebraically for $p < 1/2$ and diverging algebraically for $p > 1/2$. For $1/2 < p < 3/4$, the mean square displacement remains larger than the square of the mean, such that the behavior remains diffusive, i.e. nonpersistent. Besides the classification of the second

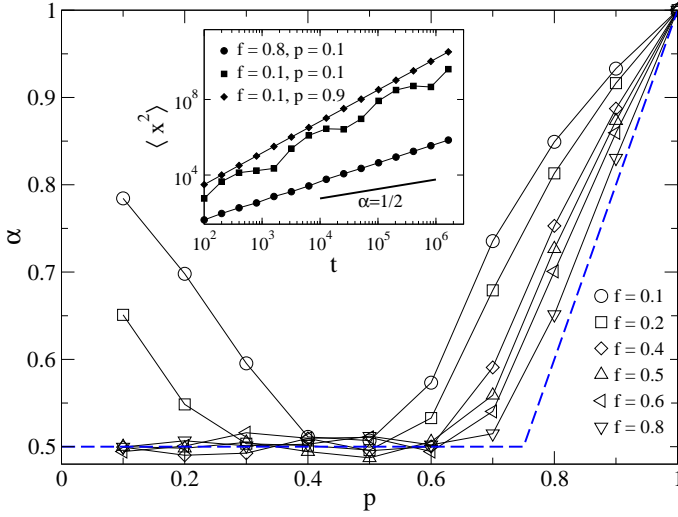


FIG. 1: Persistence of the random walk, represented by the Hurst exponent α as a function of the correlation parameter p and the fraction f of the total time remembered by non-Markovian walkers that forget the most recent $(1-f)t$ time steps, measured over 10^3 realizations. The dashed line shows the known analytic result [15, 16] for the case $f = 1$. The inset shows the typical plots of $\langle x^2 \rangle$ for chosen values of p and f from which we estimated the α . We find that whereas values $p < 1/2$ always preclude persistent behavior in the $f = 1$ case, yet persistence arises for sufficiently small f even for $p < 1/2$. Note how the $f = 0.2$ and $f = 0.1$ curves deviate away from $\alpha = 0.5$ as $p \rightarrow 0$. We find that this persistence emerges as a result of log-periodic oscillations (e.g., see the $f = p = 0.1$ case in the inset) in the velocity, such that it changes sign increasingly infrequently. Note also that loss of recent memories increases persistence for the entire range of $p \neq 1/2$.

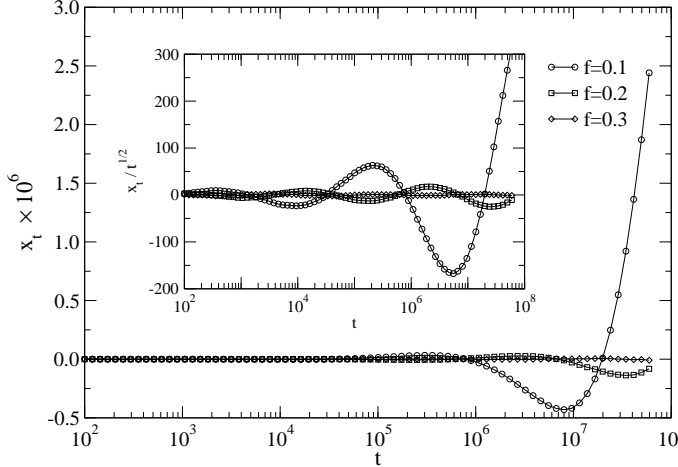


FIG. 2: Semilog plot of displacement x_t as a function of t for $p = 0.1$ and various f . The inset shows $x_t/t^{1/2}$ versus time. Significant memory loss (i.e., small f) leads to larger amplitudes for the log-periodic oscillations.

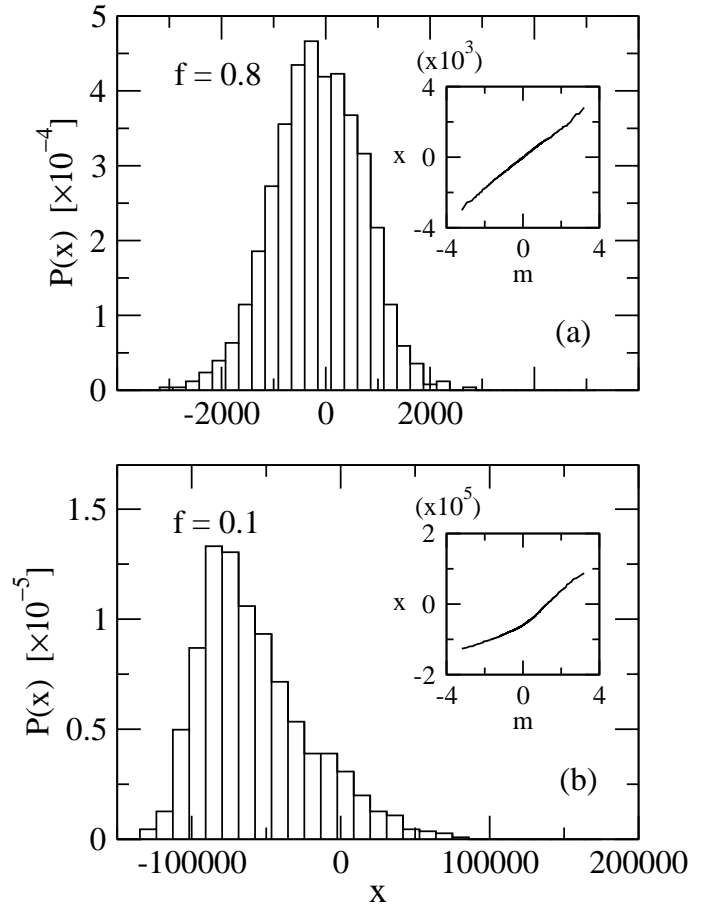


FIG. 3: Histogram showing the normalized probability density function (PDF) of the position of walkers with $p = 0.1$ after a long time $t_{max} = 1638400$, for (a) $f = 0.8$ (nonpersistent regime) and (b) $f = 0.1$ (amnestically induced persistence). We have simulated 10^3 realizations. Notice the Gaussian PDF for large f , as expected from the known properties of the $f = 1$ case. In contrast, the PDF becomes non-Gaussian for low f and low p , when memory loss induces persistence. Indeed, the aforementioned log-periodic velocity inversions (see text) allow a violation of the necessary conditions for the central limit theorem to hold, thus preventing the convergence to a Gaussian PDF. The insets show normal probability plots of the position x versus the normal order statistic medians m : the linear plot in (a) indicates Gaussian (i.e., normal) behavior, whereas the nonlinearity in (b) indicates a non-Gaussian PDF. The loss of recent memories leads to a remarkable change from Gaussian to non-Gaussian behavior—and consequently to persistence.

moment behavior in terms of nonpersistent (i.e., normal) versus persistent (i.e., anomalous) diffusion (defined by the transition at $p = 3/4$), the first moment scaling allows the classification of the random walkers as either “reformers” ($p < 1/2$) that attempt to compensate for the past behavior, or as “traditionalists” ($p > 1/2$) that tend to repeat the past. Crucially, reformers ($p < 1/2$) never show persistence ($\alpha > 1/2$). We next modify the model in order to introduce loss of memory of the recent

past.

Consider a random walker that can remember only the initial fraction ft of the total t time steps. If $f = 1$ then we recover the full memory model, but for $f < 1$ the walker, while remaining non-Markovian, nevertheless does not remember the complete history. We study how $\langle x^2 \rangle$ scales with t , as a function of p and f . Fig. 1 shows how the scaling exponent α of the mean square displacement varies as we reduce the range of the memory. As $f \rightarrow 0$ we obtain an unexpected result: even reformers ($p < 1/2$) that tend to compensate for past behavior become persistent ($\alpha > 1/2$). The loss of memory of the recent past thus appears to cause persistence for values of p for which the full memory model precludes persistence. In contrast, for loss of recall of the distant rather than recent past, the persistence can only decrease [16].

How can loss of memory lead to persistence? Note that for small f and $p < 1/2$, the random walker attempts to move opposite to the average direction chosen in the first ft time steps. It takes $t(1-f)/f$ additional time steps for the effect of the action taken at the present time t to enter into the range of the accessible memory. Therefore considerable time elapses between inversions of the time averaged velocity. The behavior thus becomes persistent, because the mean position oscillates with ever greater amplitude as $t \rightarrow \infty$, due to the increasingly infrequent velocity inversions. We have found evidence of log-periodic oscillations, indicating that discrete scale invariance, characterized by complex rather than real scaling exponents, plays a role [18]. Fig. 2 shows clear evidence of the existence of these log-periodic oscillations. The amplitude of these oscillations becomes larger for small f , whereas for sufficiently large f they effectively disappear. In the future, we expect to obtain the complete phase diagram separating the persistent and non-persistent phases as a function of p and f .

We also study a more dramatic loss of memory—the random walkers’ equivalent of anterograde amnesia: no new long-term memories can form after a simulated “accident” or “injury” at time t_a . In this case, we find ballistic ($\alpha = 1$) behavior, with no velocity inversions (or log-periodic oscillations) for any value $p \neq 1/2$.

We next report evidence that this amnestically induced persistence for $p < 1/2$ is associated with a transition from a Gaussian PDF of the position for $f = 1$ to a non-Gaussian PDF for $f < 1$ (Fig. 3). The inset in Fig. 3 shows a normal probability plot [19] of the position. On the vertical axis we plot the ordered data, and on the horizontal axis we plot the normal order statistic medians for the normal (i.e., Gaussian) distribution. Departures from a linear plot indicate departures from Gaussian statistics. We obtain a good fit with a Gaussian distribution for large f but not for small f . One would not typically expect this result, since only Gaussian PDFs have appeared in similar models [15, 16]. By changing the parameter f , the behavior undergoes a remarkable qualitative change,

from Gaussian to non-Gaussian.

These numerical results are consistent with the known analytical solutions for $f = 1$ [15] and for the case of random walkers with anterograde amnesia, which leads to a ballistic solution with constant velocity for $p \neq 1/2$. Analytic solutions for $0 < f < 1$ may require solving recurrence relations in which the moments $\langle x^q \rangle$ at time $t + 1$ depend not only on their values at a previous time t but also on the values at a time ft in the more distant past. The full analytical solution for α as a function of f and p remains an open problem. In this context, we have found preliminary evidence that the recurrence relations allow power law and log-periodic solutions in the first moment $\langle x_t \rangle$.

We also discuss the expected behavior for d -dimensional generalizations for $f < 1$. For the case of separate memories for each space direction, we expect the dimensionality not to play a major role on how α depends on f . The velocities now become d -dimensional vectors, as do the displacements. But the velocities along a given dimension do not affect the position along a distinct dimension, thereby effectively decoupling the dimensions, in accordance with the known analytical result for the $f = 1$ model [15].

We briefly comment on the above results, in the context of complex systems that have learning or self-regulating mechanisms [20] that preempt repetitive dynamics. By incorporating some form of negative feedback, many complex adaptive systems avoid persistent or repetitive behavior. For the full memory model ($f = 1$), negative feedback occurs for $p < 1/2$, so persistence cannot arise. However, for $f < 1$ the velocity inversions provoked by the negative feedback happen ever more infrequently for the reasons explained earlier, with important consequences for self-regulation. Essentially, negative feedback breaks down with the loss of recent recall, thus allowing persistence. Our results suggest the possibility of a new quantitative description of the phenomenology of memory loss and may achieve a closer connection with realistic applications. Specifically, we note (i) the possibility of quantifying the extent of memory damage in diverse (e.g., neurophysiological) non-Markovian systems via $1 - f$ and (ii) the known relationship between α and pathology [2, 13] for a number of health conditions.

We do not find it inconceivable that persistence and repetition in diverse self-regulating complex systems may emerge whenever memories of recent events degrade preferentially to those of the distant past. Consider, for example, the role of memory in health and disease [21, 22, 23]. A frequent and burdensome behavioral and psychological symptom of Alzheimer’s disease and other dementias involves persistent and repetitive behavior [23]. Patients may hum a tune that never seems to run out of verses, pose the same question dozens of times a day, or pace the same stretch of floor for hours. They may also continuously repeat words or phrases (i.e., echolalia).

Most importantly in the context of our findings, patients that suffer from persistent and continual repetition of questions (e.g., the inability to remember directions) often show clear evidence of loss of recent memory and immediate recall [23]. Memory loss frequently manifests itself in early stages of the disease while repetitive actions are less common in early dementia but increase in frequency in patients with definite diagnoses of dementia [24, 25]. While memory impairment can clearly account for repetitive questioning, its role in other repetitive actions (either as a cause or as a correlate) has remained unclear thus far [23]. In view of these facts and our reported findings, we find it plausible that recent memory loss may bear a causal relationship with the repetitive behaviors seen in Alzheimer's disease, in which memories of the distant past fade last.

In summary, we have discovered a new mechanism underlying trend-reinforcing dynamics: amnesic induction of persistence. We have shown that loss of memory of the recent past can cause persistence in otherwise non-persistent non-Markovian random walkers. Whereas the loss of distant memories decreases persistence [16], our results indicate that the loss of recent memories actually increases persistence, allowing deviations from Gaussian statistics.

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